Models of Non – Life Insurance Mathematics

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In this communication we will discuss two regression credibility models from Non – Life Insurance Mathematics that can be solved by means of matrix theory. In the first regression credibility model, starting from a well-known representation formula of the inverse for a special class of matrices a risk premium will be calculated for a contract with risk parameter θ. *In the next regression credibility model, we will obtain a credibility solution in the form of a linear combination of the individual estimate (based on the data of a particular state) and the collective estimate (based on aggregate USA data).*

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Introduction

All numerical results in this paper were obtained using the regression credibility theory. Here we consider applications of credibility theory dealing with real life situations, and implemented on real insurance portfolios.

The regression credibility model can be applied to solve quite a number of practical insurance problems.

1. The first regression credibility model

In the first regression credibility model, starting from a well-known representation formula of the inverse for a special class of matrices a risk premium will be calculated for a contract with risk parameter θ.

After some motivating introductory remarks, we state the model assumptions in more detail.

In this sense, we consider *one contract* (or *an insurance policy*) with *unknown* and *fixed* risk parameter θ, during a period of t (≥ 2) years. The random variable θ contains the risk characteristics of the policy. For this reason, we shall call θ *the risk parameter of the policy*.

The contract is a random vector $(\theta, \underset{\sim}{X}')$ consisting of the structure parameter θ and the observable variables $X_1, X_2, \ldots, X_t$, where $\underset{\sim}{X}' = (X_1, X_2, \ldots, X_t)$ is the vector of observations (or the observed random $(1 \times t)$ vector). Thus, the contract consists of the set of variables:

$$(\theta, \underset{\sim}{X}') = \theta, X_j, \text{ where } j = 1, \ldots, t$$

For the model, which involves only one isolated contract and having observed a risk with risk parameter θ for t years we want to *forecast/estimate* the quantity (the conditional expectation of the X_j, given θ):

$$\mu_j(\theta) = E(X_j \mid \theta)$$

which is *the net risk premim* for *the contract* with *risk parameter* θ from *the j-th year*, where

$j = 1, \ldots, t$. Because of inflation, we make *the regression assumption,* which affirms that the pure net risk premium $\mu_j(\theta)$ changes in time, as follows:

$$\mu_j(\theta) = E(X_j \mid \theta) = \underset{\sim}{Y}'_j \underset{\sim}{b}(\theta), \; j = 1, \ldots, t,$$

where $\underset{\sim}{Y}_j$ is an known non-random $(q \times 1)$ vector, the so-called *design vector,* with $j = 1, \ldots, t$ and where $\underset{\sim}{b}(\theta)$ is an unknown random $(q \times 1)$ vector, the so-called *regression vector,* which contains the unknown regression constants.

By a suitable choice of the $\underset{\sim}{Y}_j$ (assumed to be known), time effects on the risk premium can be introduced.

Thus, if the design vector $\underset{\sim}{Y_j}$ is for example chosen as follows:

$\underset{\sim}{Y_j} = \underset{\sim}{Y_j}^{(2,1)} = \begin{pmatrix} 1 \\ j \end{pmatrix}$, then results a *linear inflation* of the type:

$\mu_j(\theta) = b_1(\theta) + jb_2(\theta)$, j = 1, …, t, where $\underset{\sim}{b}(\theta) = (b_1(\theta), b_2(\theta))'$.

Also if the design vector $\underset{\sim}{Y_j}$ is for example chosen as follows:

$\underset{\sim}{Y_j} = \underset{\sim}{Y_j}^{(3,1)} = \begin{pmatrix} 1 \\ j \\ j^2 \end{pmatrix}$, then we obtain a *quadratic inflationary trend* of the form:

$\mu_j(\theta) = b_1(\theta) + jb_2(\theta) + j^2b_3(\theta)$, j = 1, …, t, where $\underset{\sim}{b}(\theta) = (b_1(\theta), b_2(\theta), b_3(\theta))'$.

Using the fact that a matrix A is *positive definite,* if the quadratic form $\underline{x}'A\underline{x}$ is positive for every $\underline{x} \neq \underline{0}$, where A is an (n × n) matrix, $\underline{x}$ a column vector of length n and $\underline{0}$ is a vector of zeros, we can give the hypotheses of the model. We assume that:

(1) the *regression assumption,* which affirms that the pure net risk premium $\mu_j(\theta)$ for the contract with risk parameter θ from j-th year changes in time, as follows:

$\mu_j(\theta) = \underset{\sim}{Y_j}'\underset{\sim}{b}(\theta)$, j = 1, …, t, where the (q × 1) design vector $\underset{\sim}{Y_j}$ is known, with j = 1, …, t and $\underset{\sim}{b}(\theta)$ is an unknown regression vector ($\underset{\sim}{b}(\theta)$ is a column vector of length q) and that

(2) the matrices $\underset{\sim}{\Lambda} = \underset{\sim}{\Lambda}^{(q,\,q)} = \mathrm{Cov}[\underset{\sim}{b}(\theta)]$, $\underset{\sim}{\phi} = \underset{\sim}{\phi}^{(t,\,t)} = E[\mathrm{Cov}(\underset{\sim}{X} \mid \theta)]$ are *positive definite* [$\underset{\sim}{\Lambda}$ is the covariance matrix of the regression vector $\underset{\sim}{b}(\theta)$, and $\underset{\sim}{\phi}$ is the expectation for the conditional covariance matrix of the observations $\underset{\sim}{X}$, given θ].

The *main* purpose of regression credibility theory is the development of an expression for the credibility estimator $\underset{\sim}{\tilde{\mu}_j}$ of the pure net risk premium $\mu_j(\theta)$ based on the observations $\underset{\sim}{X}$.

For this reason, we shall need (we need) the following *lemma* from linear algebra, which gives the *representation formula of the inverse for a special class of matrices.*

Lemma 1.1 Let $\underset{\sim}{A}$ be an (r × s) matrix and $\underset{\sim}{B}$ an (s × r) matrix. Then the inverse of the matrix $(\underset{\sim}{I} + \underset{\sim}{A}\,\underset{\sim}{B})$ is given by the formula:

$(\underset{\sim}{I} + \underset{\sim}{A}\,\underset{\sim}{B})^{-1} = \underset{\sim}{I} - \underset{\sim}{A}(\underset{\sim}{I} + \underset{\sim}{B}\,\underset{\sim}{A})^{-1}\underset{\sim}{B}$, if the displayed inverses exist and where $\underset{\sim}{I}$ denotes the (r × r) identity matrix.

We finally introduce the following *notation* for the expectation of the regression vector $E[\underset{\sim}{b}(\theta)] = \underset{\sim}{\beta}$.

Now, we are ready to determine the *optimal choice* of the *credibility estimator* $\tilde{\mu}_j$ for the pure net risk premium $\mu_j(\theta)$ based on the observations $\underset{\sim}{X}$.

Under the hypotheses **(1)** and **(2)** the credibility estimator $\tilde{\mu}_j$ for the pure net risk premium $\mu_j(\theta)$ based on the observations $\underset{\sim}{X}$ is given by the following relation:

$$\tilde{\mu}_j = \underset{\sim}{Y_j}'[\underset{\sim}{Z}\,\underset{\sim}{\hat{b}} + (\underset{\sim}{I} - \underset{\sim}{Z})\underset{\sim}{\beta}] \text{ with}$$

$$\underset{\sim}{\hat{b}} = (\underset{\sim}{Y}'\,\underset{\sim}{\phi}^{-1}\,\underset{\sim}{Y})^{-1}\,\underset{\sim}{Y}'\,\underset{\sim}{\phi}^{-1}\,\underset{\sim}{X} \text{ and}$$

$$\underset{\sim}{Z} = \underset{\sim}{\Lambda}\,\underset{\sim}{Y}'\,\underset{\sim}{\phi}^{-1}\,\underset{\sim}{Y}\,(\underset{\sim}{I} + \underset{\sim}{\Lambda}\,\underset{\sim}{Y}'\,\underset{\sim}{\phi}^{-1}\,\underset{\sim}{Y})^{-1},$$

where $\underset{\sim}{Y}$ is *the generalization* of the design vector $\underset{\sim}{Y_j}$, the so-called *design matrix* from the *regression assumption* (1) of the type: $\underset{\sim}{\mu}^{(t,1)} = E(\underset{\sim}{X} \mid \theta) = \underset{\sim}{Y}\,\underset{\sim}{b}(\theta)$ and where $\underset{\sim}{I}$ denotes the (q × q) identity matrix, for some fixed j. [$\underset{\sim}{\mu}^{(t,1)} = (\mu_1(\theta), \mu_2(\theta), \ldots, \mu_t(\theta))'$ is the (t × 1) vector of the yearly net risk premiums for the contract with risk parameter θ and $\underset{\sim}{Y}$

is an (t × q) matrix given in advance of full rank q (q ≤ t)].
We recall the fact that a matrix A is of *full rank* if its rank is *min(n, m),* where A is an (n × m) matrix.
Now, we give the proof of the above expression for the credibility estimator $\tilde{\mu}_j$ of the pure net risk premium $\mu_j(\theta)$ based on the observations $\underset{\sim}{X}$. The credibility estimator $\tilde{\mu}_j$ of $\mu_j(\theta)$ based on $\underset{\sim}{X}$ is a linear estimator of the form:

$$\tilde{\mu}_j = \gamma_0 + \underset{\sim}{\gamma}' \underset{\sim}{X} \qquad (1.1),$$

which satisfies the normal equations:

$$E(\tilde{\mu}_j) = E[\mu_j(\theta)] \qquad (1.2)$$

$$Cov(\tilde{\mu}_j, \underset{\sim}{X}') = Cov[\mu_j(\theta), \underset{\sim}{X}'] \qquad (1.3),$$

where γ_0 is a scalar constant and $\underset{\sim}{\gamma}$ is a constant (t × 1) vector. The coefficients γ_0 and $\underset{\sim}{\gamma}$ are chosen such that the normal equations are satisfied.
After inserting (1.1) in (1.3), one obtains the following relation:

$$\underset{\sim}{\gamma}' Cov(\underset{\sim}{X}) = Cov[\mu_j(\theta), \underset{\sim}{X}'] \quad (1.4),$$

where:

$$Cov(\underset{\sim}{X}) = \underset{\sim}{\phi} + \underset{\sim}{Y}\, \underset{\sim}{\Lambda}\, \underset{\sim}{Y}' \qquad (1.5)$$

and:

$$Cov[\mu_j(\theta), \underset{\sim}{X}'] = \underset{\sim}{Y}'_j\, \underset{\sim}{\Lambda}\, \underset{\sim}{Y}' \qquad (1.6).$$

Standard computations lead to (1.5) and (1.6). Thus, (1.4) becomes:

$$\underset{\sim}{\gamma}'(\underset{\sim}{\phi} + \underset{\sim}{Y}\, \underset{\sim}{\Lambda}\, \underset{\sim}{Y}') = \underset{\sim}{Y}_j'\, \underset{\sim}{\Lambda}\, \underset{\sim}{Y}'$$

Hence, applying Lemma 1 we conclude that:

$$\underset{\sim}{\gamma}' \underset{\sim}{X} = \underset{\sim}{Y}'_j\, \underset{\sim}{Z}\, \underset{\sim}{\hat{b}} \qquad (1.7)$$

From (1.1) (1.2) and (1.7) we obtain:

$$\gamma_0 = \underset{\sim}{Y}'_j (\underset{\sim}{I} - \underset{\sim}{Z}) \underset{\sim}{\beta}$$

This completes the proof.

2. The second regression credibility model

In the next regression credibility model, we will obtain a credibility solution in the form of a linear combination of the individual estimate (based on the data of a particular state) and the collective estimate (based on aggregate USA data).
To illustrate the solution with the properties mentioned above, we shall need the well-known representation theorem for a special class of matrices, the properties of the trace for a square matrix, the scalar product of two vectors, the norm with respect to a positive definite matrix given in advance and the complicated mathematical properties of conditional expectations and of conditional covariances.
After some motivating introductory remarks, we state the model assumptions in more detail.
In this sense, we consider a portfolio of k contracts. Let j be fixed.
The contract indexed by j is a random vector $(\theta_j, \underline{X}'_j)$ consisting of a random structure parameter θ_j (assumed to be *unknown* and *fixed*) and the observable variables X_{j1}, X_{j2}, …, X_{jt}, where $\underline{X}'_j = (X_{j1}, X_{j2}, \ldots, X_{jt})$ is the vector of observations (or the observed random (1 × t) vector). So the contract indexed by j consists of the set of variables:
$(\theta_j, \underline{X}'_j) = \theta_j, X_{jq}, q = 1, \ldots, t$
For the model, which consists of a portfolio of k contracts we want to *forecast/estimate* the quantity (the conditional expectation of the X_{jq}, given θ_j):

$$\mu_q(\theta_j) = E(X_{jq} \mid \theta_j),\ q = 1, \ldots, t$$

(which is the net risk premium for the contract with risk parameter θ_j from the q-th year, where q = 1, …, t), or we want to *forecast/estimate* the conditional expectation of the $\underline{X}_j$, given θ_j:

$$E(\underline{X}_j \mid \theta_j) = \underline{\mu}^{(t,1)}(\theta_j) = (\mu_1(\theta_j), \ldots, \mu_t(\theta_j))'$$

(which is the vector of the yearly net risk premiums for the contract with risk parameter θ_j).
Because of inflation, we make the *regression assumption* (or we restrict the class of admissible functions $\mu_q(\cdot)$ to):

$$a^* = \frac{1}{w_\cdot^2 - \sum w_j^2}\left\{\frac{1}{2}\sum_{i,j} w_i w_j (\underline{B}_i - \underline{B}_j)(\underline{B}_i - \underline{B}_j)' - \hat{s}^2 \sum_{j=1}^{k} w_j (w. - w_j) u_j\right\}$$

Conclusions

The matrix theory provided the means to calculate useful estimators for the structure parameters.

From the practical point of view the property of unbiasedness of these estimators is very appealing and very attractive.

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