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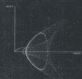
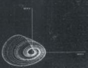
NUMERICAL MODELING AND SIMULATION OF DYNAMICAL SYSTEMS





$$\dot{x} = F(u,x) \rightarrow x_k = g \cdot F_k + h_{xFk}$$

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THEORETICAL PRELIMINARIES FOR THE LOCAL-ITERATIVE LINEARIZATION METHOD

Numerous scientifical and technological dynamical processes can be analytically modeled through the differential equations, which in the normal form are:

$$\frac{d^n y}{dt^n} = F\left(u, \frac{du}{dt}, \ldots, \frac{d^m u}{dt^m}, y, \frac{dy}{dt}, \ldots, \frac{d^{n-1} y}{dt^{n-1}}\right). \tag{1.1}$$

The input signal $u=u(t)$ as independent, known variable, the output signal $y=y(t)$ as dependent, unknown variable, and the right member of the equation (1.1) are continuously differentiable functions.

The initial conditions are considered known, and always $n \geq m$ with $n \geq 1$ in the real situations.

For the numerical integrations of the equations (1.1) we will consider an integrand developed in a Taylor series:

$$\Psi(t) = \Psi(t_k + \theta) = \Psi_k(\theta) = \sum_{q=0}^{\infty} \frac{\theta^q}{q!} \left(\frac{d^q \Psi}{dt^q}\right)_k, \tag{1.2}$$

where $t_k = k \cdot \Delta t$ represents the pivot moment at the sequence k and $\Delta t = 2 \cdot \delta t$ is the integration step, considered small enough. The index k is used here for the functions specification at the moment t_k.

The analytical resolution of (1.2) leads to

$$\int_{t_k-\delta t}^{t_k+\delta t} \underset{n}{\int \ldots \int} \Psi(t)\, dt^n = \int_{-\delta t}^{\delta t} \underset{n}{\int \ldots \int} \Psi_k(\theta)\, d\theta^n =$$

$$= \sum_{q=0}^{\infty} \left[\frac{\theta^{n+q}}{(n+q)!} + \sum_{\lambda=1}^{n-1} (-1)^{\lambda} \frac{(-\delta t)^{q+\lambda}}{(\lambda-1)!\, q!\, (q+\lambda)} \frac{\theta^{n-\lambda}}{(n-\lambda)!} \right]_{\theta=-\delta t}^{\theta=\delta t} \Psi_k^{(q)}, \tag{1.3}$$

where the integration constants are calculated at the moment of beginning of integration, i.e. at $t = t_k - \delta t$, respectively at $\theta = -\delta t$ and we denoted $\Psi_k^{(q)} = (d^q \Psi / dt^q)_k$.

This derivative $\Psi_k^{(q)}$ is numerically approximated by the usual form

$$\Psi_k^{(q)} = \frac{1}{\Delta t^q}\sum_{j=0}^{\omega}\delta_{qj\omega}\Psi_{k-j}, \tag{1.4}$$

where $\delta_{qj\omega}$ -the weighting coefficients are fraction expressions which depend on the order q, the sequence j and the last regressive sequence ω. We have denoted $\Psi_{k-j} = \Psi(t_{k-j}) = \Psi(t_k - j\cdot\Delta t)$. If $\Psi(t)$ has the form

$$\Psi(t) = F\frac{d^p v}{dt^p} = Fv^{(p)}, \tag{1.5}$$

there for F constant,(1.3) becomes

$$\int_{t_k-\Delta t}^{t_k}\!\!{}_{n}\int Fv^{(p)}dt^n = F\Delta t^{n-p}\sum_{j=0}^{\omega}\sigma_{npj\omega}\cdot v_{k-j} + \rho_{np\omega}(\Delta t^r). \tag{1.6}$$

Here we have used the following notations: $\sigma_{npj\omega}$ for the weighting coefficients, which depend on n,p,j,ω, and $v_{k-j} = v(t_{k-j}) = v(t_k - j\cdot\Delta t)$ for the regressive sequences (j=1,2,...,ω) or the current sequence (j=0) of the variable v(t). This variable may be an input signal u(t), an output signal y(t), or a state one x(t). The total error of the approximation, due to Taylor series cut off, is expressed by $\rho_{np\omega}(\Delta t^r)$, where $r = n + \omega + 1 - p$. Then the order of magnitude of the error depends on n,p and ω, where ω is the last regressive sequence take into account, with the condition that $\omega \geq n$ and p=0,1,2,...,n.

In the Table 1 we present the values of the weighting coefficients $\delta_{qj\omega}$ for q=1,2,...5; j=0,1,...,5 and $\sigma_{npj\omega}$ for n=1, p=0,1 and j=0,1,...,ω. The last regressive sequence ω taken into account for $\delta_{qj\omega}$, $\sigma_{npj\omega}$ and $\rho_{np\omega}$ are ω=3 and ω=5.

Table 1

q	ω	$\delta_{q0\omega}$	$\delta_{q1\omega}$	$\delta_{q2\omega}$	$\delta_{q3\omega}$	$\delta_{q4\omega}$	$\delta_{q5\omega}$
1	3	11/6	-18/6	9/6	-2/6	-	-
	5	137/60	-300/60	300/60	-200/60	75/60	-12/60
2	3	2	-5	4	-1	-	-
	5	45/12	-154/12	214/12	-156/12	61/12	-10/12
3	3	1	-3	3	-1	-	-
	5	17/4	-71/4	118/4	-98/4	41/4	-7/4
4	5	3	-14	26	-24	11	-2
5	5	1	-5	10	-10	5	-1

p	ω	$\sigma_{p0\omega}$	$\sigma_{p1\omega}$	$\sigma_{p2\omega}$	$\sigma_{p3\omega}$	$\sigma_{p4\omega}$	$\sigma_{p5\omega}$
0	3	13/12	-5/24	1/6	-1/24	-	-
	5	741/640	-1561/2880	2179/2880	-133/240	1253/5760	-103/2880
1	3	15/8	-25/8	13/8	-3/8	-	-
	5	315/128	-735/128	399/64	-279/64	215/128	-35/128

ω	$e_{01\omega}$	$e_{02\omega}$	$e_{03\omega}$	$e_{04\omega}$	$e_{05\omega}$
3	3	-3	1	-	-
5	5	-10	10	-5	1

The extrapolation coefficients $e_{\varphi\omega}$ indicated in the same table, are need for the approximation forms:

$$v_k = v_{k\omega} - \sum_{j=1}^{\omega} e_{j\omega} v_{k-j}, \qquad (1.7)$$

which will be used in the following.

We note that the numerical approximation (1.6), which will be used in the L.I.I. technique, has errors $\rho_{\varphi p\omega}(\Delta t^{\omega})$ small enough. This may be explained by the fact that due to the symmetrical integration with respect to t_k in (1.6), a half of terms from those neglected in Taylor series are in fact eliminated (see 3.25 and 3.26). In the following, for the simplicity, we will neglect the last index ω, and we will use the notations: $\delta_{\varphi p}$, $\sigma_{\varphi pj}$ and e_{j}

NUMERICAL MODELLING THROUGH L.I.L. WITH INPUT-OUTPUT RELATIONS

Let us return to the analytical model(1.1), expressed by the input-output relations, in the simplified form:

$$y^{(q)}+F(u^{(q)},y^{(p)})=F, \quad (q=0,...,m;p=0,...,n-1) \tag{2.1}$$

which is numerically integrated by using (1.6), i.e.

$$\int_{t_k-\Delta t}^{t_k+\Delta t}{}^{(n)}\!\!\int y^{(q)}dt^{n}=\int_{t_k-\Delta t}^{t_k+\Delta t}{}^{(n)}\!\!\int F dt^{n} \tag{2.2}$$

from which we obtain

$$\sum_{j=0}^{n}\alpha_{nqj}y_{k-j}=\Delta t^{n}\sum_{j=0}^{n}\alpha_{n0j}F_{k-j} \tag{2.3}$$

Then we find

$$y_k=\Delta t^{n}\frac{\alpha_{n00}}{\alpha_{nn0}}F_k-\frac{1}{\alpha_{nn0}}\sum_{j=1}^{n}(\Delta t^{n}\alpha_{n0j}F_{k-j}-\alpha_{nnj}y_{k-j})=gF_k+b_k \tag{2.4}$$

where

$$g=\Delta t^{n}\frac{\alpha_{n00}}{\alpha_{nn0}} \tag{2.5}$$

is the transfer coefficient

The term $g \cdot F_k$ can be considered as the **forced component** of the numerical solution y_k. We observe that

$$F_k = F(u_k^{(p)}, y_k^{(p)}) = F(u_k^{(p)}, y_{ke}^{(p)}), \tag{2.6}$$

where

$$y_{ke}^{(p)} = \sum_{j=1}^{w} \epsilon_{kj} \cdot y_{k-j}^{(p)} \tag{2.7}$$

represents the extrapolated expression which has the same form as (1.7)

The term

$$h_k = \frac{1}{\alpha_{m0}} \sum_{j=1}^{w} (\Delta t^n a_{nj} F_{k-j} - \alpha_{mj} y_{k-j}) \tag{2.8}$$

corresponds to the **free component** of the numerical solution y_k

This represents "the history of the process" because it contains only the regressive sequences y_{k-j} and F_{k-j} for $j = 1, 2, \ldots, w$.

The steps of the calculus (2.1)-(2.8) may be also applied to the particular case of the ordinary differential equations

$$\sum_{p=0}^{n} a_p \frac{d^p y}{dt^p} = \sum_{p=0}^{m} b_p \frac{d^p u}{dt^p}, \; (n \ge m) \tag{2.9}$$

where a_p and b_p could be constant coefficients or continuous in time functions. Then according to (1.6) we have

$$\int_{t_k - \Delta t}^{t_k + \Delta t} \beta \int a_p \frac{d^p y}{dt^p} = \int_{t_k - \Delta t}^{t_k + \Delta t} \beta \int \sum_{p=0}^{m} b_p \frac{d^p u}{dt^p} dt^n, \tag{2.10}$$

which leads to

$$\sum_{j=0}^{w} \sum_{p=0}^{n} \Delta t^{n-p} a_p \beta_{mj} y_{k-j} = \sum_{j=0}^{w} \sum_{p=0}^{m} \Delta t^{n-p} b_p \beta_{mj} u_{k-j}, \tag{2.11}$$

where to simplify the expressions, the coefficients a_p and b_p are considered constat.

As a result we obtain the numerical solution, local-iterative linearizated

$$y_k = g \cdot u_k + h_k, \tag{2.12}$$

where

$$g = \frac{\sum_{p=0}^{m} \Delta t^{n-p} b_p \sigma_{np0}}{\sum_{p=0}^{n} \Delta t^{n-p} a_p \sigma_{np0}}, \tag{2.13}$$

and

$$h_k = \frac{\sum_{j=1}^{m}\sum_{p=0}^{m} \Delta t^{n-p} b_p \sigma_{npj} u_{k-j} - \sum_{j=1}^{n}\sum_{p=0}^{n} \Delta t^{n-p} a_p \sigma_{npj} y_{k-j}}{\sum_{p=0}^{n} \Delta t^{n-p} a_p \sigma_{np0}}, \tag{2.14}$$

We note that (2.13) and (2.14) may be interpreted in the same way as (2.5) and (2.8) respectively.

The form (2.12) has many advantages in dynamical systems analysis and synthesis because it contains analytically modeled by (2.9) elements. One of these advantages is exemplified in the figure 2.1, where we can see the possibility to give an algebra to the three fundamental connections - series, parallel and with feedback - to obtain the equivalent expressions of g_e and h_e.

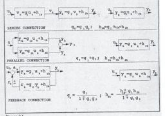

Figure 2.1

We can observe that g_e has the expression of the equivalent transfer function for the same connections, and h_{e} can be obtained by simple rules.

The LIL modelling with input-output relations has also the other advantages in the theory of systems, but we do not continue our study in this direction. The same principles presented in this chapter can be applied in the case of the input-state relations. As a result we can develop an efficient, simple and unitary method of modelling and simulation by LIL.