Fişa suspiciunii de plagiat / Sheet of plagiarism's suspicion	Indexat la: 53/05

Opera suspicionată (OS)	Opera autentică (OA)
Suspicious work	Authentic work

OS	Ciobanu C., About relationship between generalized structurable algebras and Lie related triples, In: An.St. Univ. Ovidius Constanta, Vol. 15(1), 2007, p.47–54.
OA	Kamiya N., On generalized structurable algebras and Lie related triples, In: Advances in Applied Clifford Algebras 5, No. 2 (1995), p.127-140.

Incidența minimă a suspiciunii / Minimum incidence of suspicion	
p.47:20 – p.47:22	p.127:28 – p.127:31
p.48:2 - p.48:31; p.49:1 – p.49:5	p.127:33 – p.127:36; p.128:1 – p.128:25; p.129:1 – p.129:6
p.49:19 – p.49:31	p.130: 14 – p.130:23
p.49:12 – p.49:28	p.131: 24 – p.131:28; p.132:1 - p.132:15
p.51:3 – p.51:17	p.132: 18 – p.132:28; p.133:1 – p.133:5
p.51:19 – p.51:25	p.133: 7 - p.133:12
p.51:26 – p.51:30; p.52:1 – p.52:4	p.135:3 – p.135:10
p.52:5 – p.52:30; p.53:1 – p.53:9	p.135:11 - p.135:25; p.136:1 - p.136:21

ON GENERALIZED STRUCTURABLE ALGEBRAS AND LIE RELATED TRIPLES

Noriaki Kamiya
Department of Mathematics,
Faculty of Science,
Shimane University,
MATSUE, 690, JAPAN

Abstract

In this paper, we give examples of generalized structurable algebras and investigate their standard embedding generalized structurable algebras.

Introduction

Recently, we have investigated generalized structurable algebras ([3],[4], [5]). These generalized structurable algebras contain the classes of Clifford algebras, Lie algebras, alternative algebras, Poisson algebras, and a class of nonassociative algebras appearing in mathematical physics. That is, broadly speaking, it seems that our algebras are valuable in characterizing physically relevant phenomenon.

It is one of our aims to give examples of generalized structurable algebras in this paper.

The generalized structurable algebras and their standard embedding structurable algebras are considered in Section 1.

In Section 2, we shall introduce the Lie related triple due to Prof. N. Jacobson.

In Section 3, we will investigate the relationship between generalized structuralbe algebras and Lie related triples.

In Section 4, examples of the standard embedding generalized structurable algebras associated with generalized structurable algebras will be presented.

We shall be concerned with algebras and triple systems which are finite or infinte dimensional over a commutative associative ring of scalars Φ without 2–torsion or 3–torsion (i.e, $2x = 0$ or $3x = 0 \Rightarrow x = 0$), unless otherwise specified.

1. Definition and Preliminaries

In this section, we shall exhibit several examples of generalized structurable algebras. First we define a *generalized structurable algebra* over a ring of scalars Φ to be a nonassociative algebra A equipped with a bilinear derivation $D(x, y)(\neq 0)$ for which the following conditions are satisfied

$$D(x,y) = -D(y,x) \tag{1-1}$$

$$D(xy,z) + D(yz,x) + D(zx,y) = 0 \tag{1-2}$$

for all $x,y,z \in A$.

Example 1. Let $(A, [\ ,\ ])$ be a Lie algebra over Φ. Then A is a generalized structurable algebra equipped with a derivation

$$D(x,y) := ad[x,y].$$

Example 2. Let $(A, \circ)$ be a commutative Jordan algebra over Φ. Then A is a generalized structurable algebra equipped with a derivation

$$D(x,y) := [L(x), L(y)],$$

where

$$[L(x), L(y)]z = x \circ (y \circ z) - y \circ (x \circ z), L(x)z = x \circ z.$$

Example 3. Let A be a Clifford algebra over Φ. Then A is a generalized structurable algebra equipped with a derivation

$$D(x,y) := [L(x), L(y)] + [L(x), R(y)] + [R(x), R(y)]. \tag{1-3}$$

In fact, a Clifford algebra is an associative algebra. From the result of [3], it follows that the associative algebras are contained in a class of a generalized structurable algebras equipped with the above derivation (1–3).

Example 4. Suppose $(A, ^-)$ is a unital algebra with an involution $^-$ over a field Φ of $ch\Phi \neq 2, 3$. Then $(A, ^-)$ is said to be *structurable* if

$$[L(x,y), L(z,w)] = L(< xyz >, w) - L(z, < yxw >),$$

where

$$L(x,y)z =< xyz >:= (x\bar{y})z + (z\bar{y})x - (z\bar{x})y$$

for x,y,z in A (cf. [1]).
Then A is a generalized structurable algebra equipped with a derivation

$$D(x,y) := \frac{1}{3}([[x,y] + [\bar{x},\bar{y}], z]) + [z,y,x] - [z,\bar{x},\bar{y}], \tag{1-4}$$

where

$$[x, y] := xy - yx, [x, y, z] := (xy)z - x(yz).$$

In fact, the identities (1–1) and (1–2) are valid for the structurable algebra A. Thus any structurable algebra is a generalized structurable algebra. The derivation $D(x, y)$ induced by (1–4) is said to be an *inner derivation* of the structurable algebra. The set of all such derivation is denoted by InnDer A.

From this point on, for a generalized structurable algebra A equipped with a derivation $D(x, y)$, we also denote by InnDer A, the subspace spanned by all the $D(x, y)$.

Theorem 1.1. **([3][4])** *Let A be a generalized structurable algebra over Φ equipped with a derivation $D(x, y)$ satisfying* $\sum_{(x,y,z)}(D(x, y)z + [[x, y], z]) = 0$ *and* $[D(z, w), D(x, y)] = D(D(z, w)x, y) + D(x, D(z, w)y) for$ $x, y, z, w \in A$. *Then the vector space* $L(A) :=$ InnDer $A \oplus A$ *is a Lie algebra with respect to the new bracket operation* $[\ ,\ ]^*$

$$[D + x, E + y]^* := [D, E] + D(x, y) + Dy - Ex + [x, y] \qquad (1-5)$$

$$\text{for } D, E \in \text{InnDer } A,\ \ x, y \in A.$$

The Lie algebra obtained from this theorem is said to be the *standard embedding Lie algebra* associated with the generalized structurable algebra.

Remark ([3]). Let us we consider the following cases of Lie algebra construction:
(i) the first Tit's construction,
(ii) the second Tit's construction.
Then we note that the above two constructions of Lie algebras may be characterized by means of our concept (Theorem 1.1) via the generalized structurable algebras.

Remark ([4]). Let $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ be a decomposition into the subspace L_i of a Lie algebra L satisfying

$$[L_i, L_j] \subset L_{i+j}, \qquad (if\ |i+j| \leq 2)$$

$$[L_i, L_j] = 0, \qquad (if\ |i+j| \geq 3)$$

$$\sum_{i \neq 0} [L_i, L_{-i}] = L_0$$

for all integers i, j satisfying $-2 \leq i, j \leq 2$.

Then $A = L_{-2} \oplus L_{-1} \oplus L_1 \oplus L_2$ is a generalized structurable algebra with respect to the product

$$X \circ Y := [x_{-2}, y_1] + [x_{-1}, y_{-1}] + [x_{-1}, y_2] +$$

$$[x_1, y_{-2}] + [x_1, y_1] + [x_2, y_{-1}]$$

and the derivation

$$D(X, Y) := ad([x_{-2}, y_2] + [x_{-1}, y_1] + [x_1, y_{-1}] + [x_2, y_{-2}])$$

$$X = x_{-2} + x_{-1} + x_1 + x_2, \quad Y = y_{-2} + y_{-1} + y_1 + y_2$$

for $x_i, y_i \in L_i$.

As a generalization of Theorem 1.1, we have the following theorem.

Theorem 1.2. (Extended property for generalized structurable algebra) *Let A be a generalized structurable algebra over Φ equipped with a derivation $D(x, y)$ such that the derivations satisfy the relation $[D, D(x, y)] = D(Dx, y) + D(x, Dy)$, for all D, $D(x, y) \in \mathrm{InnDer}\ A$. Then the vector space $B := \mathrm{InnDer}\ A \oplus A$ is a generalized structurable algebra equipped with a new product and new derivation defined by*

$$[X, Y]^* := D_1 x_2 - D_2 x_1 + [x_1, x_2] + [D_1, D_2] + D(x_1, x_2),$$

$$D^*(X, Y)Z :=$$

$$[[D_1, D_2], D_3] + [D(x_1, x_2), D_3] + D(x_1, x_2)x_3 + [D_1, D_2]x_3$$

for $X = D_1 + x_1, Y = D_2 + x_2, Z = D_3 + x_3, D_i \in \mathrm{InnDer}\ A,\ x_i \in A$.

Proof. From the definitions of the product $[\ ,\ ]^*$ and the derivation $D(X,Y)$, it is enough to show that the following identities hold:

$$D^*(X,Y) = -D^*(Y,X),$$

$$\sum_{(X,Y,Z)} D^*([X,Y]^*, Z)W = 0,$$

$$D^*(X,Y)[Z,W]^* = [D^*(X,Y)Z, W] + [Z, D^*(X,Y)W].$$

These identities are verified by straightforward calculations. However, since they are lengthy, we omit them. This completes the proof.

The generalized structurable algebra obtained from Theorem 1.2 is said to be the standard embedding generalized structurable algebra associated with the generalized structurable algebra.

Remark. We note that the results in this section can be generalized to super or graded concepts (for example, see [3],[4],[5]).

2. Lie Related Triples

In this section, we shall recall some concepts related to triality in octonion algebras as developed in [2].

Let $(A, -)$ be any nonassociative algebra with involution over a ring of scalars Φ. Denote by $gl(A)$ the Lie algebra of Φ–endomorphisms of A. If $\mathbf{A} \in gl(A)$, define $\bar{\mathbf{A}}$ by $\bar{\mathbf{A}}(x) = (\mathbf{A}(\bar{x}))^-$.
We say that

$$\mathbf{T} = (T_1, T_2, T_3) \in gl(A)^{(3)} := gl(A) \oplus gl(A) \oplus gl(A)$$

is a *partially Lie related triple product such that*

$$\bar{T}_1(ab) = T_2(a)b + aT_3(b), \qquad (2-1)$$

for all $a, b \in A$, *and we denote the set of all partially Lie related triples by* J_o.

Example ([7]). (Principle of triality) *Let* $\mathbf{O}$ *be a Cayley algebra over a field of characteristic* $\neq 2, 3$ *with norm* $n(x)$, *and let* $o(8,n)$ *be the orthogonal Lie algebra of all* T *in End(*$\mathbf{O}$*) which are skew relative to* $n(x)$. *Then for every* T_1 *in* $o(8,n)$, *there are unique* T_2, T_3 *in* $o(8,n)$ *satisfying*

$$T_1(xy) = (T_2(x)y + x(T_3(y))$$

for all $x, y \in \mathbf{O}$.

We also say that $\mathbf{T} = (T_1, T_2, T_3)$ is a *Lie related triple* if

$$(T_i, T_j, T_k) \in J_o \qquad (2-2)$$

for all (i, j, k) which are cyclic permutations of (1,2,3).

The set of Lie related triples is denoted by J. It is easy to show that both J_o and J are Lie subalgebras of $gl(A)^{(3)}$.

A paricular triple that will be of importance to us is $\mathbf{T} = (T_1, T_2, T_3)$, where

$$T_i = L_{\bar{b}} L_a - L_{\bar{a}} L_b$$

$$T_j = R_{\bar{b}} R_a - R_{\bar{a}} R_b \qquad (2-3)$$

$$T_k = R_{(\bar{a}b - \bar{b}a)} + L_b L_{\bar{a}} - L_a L_{\bar{b}},$$

where (i, j, k) is a cyclic permutation of (1,2,3), and $a, b \in A$, with $L_a(b) = ab = R_b(a)$.

We denote the Φ–span of all such triples by J_I and say that $T \in J_I$ is an *inner triple* of the algebra $(A, -)$.

We shall discuss the example of the Lie related triples in Section 3.

3.

In this section, we shall investigate the construction of a generalized structurable algebra from a generalized structurable algebra via the standard embedding as described in Section 1.

Let A be a nonassociative algebra with involution over a ring of scalars Φ and $A[ij], 1 \leq i \neq j \leq 3$, be a copy of A, with

$$a[ij] = -\gamma_i \gamma_j^{-1} \bar{a}[ji], \gamma_i, \gamma_j \in \Phi^*.$$

We form an algebra

$$B := A[12] \oplus A[23] \oplus A[31]$$

defined by

$$[a[ij], b[jk]] = -[b[jk], a[ij]] = ab[ik] \qquad (3-1)$$

for distinct i, j, k, and all other products are identically zero.

Then we put

$$D(a[ij], b[ij]) := \gamma_i \gamma_j^{-1} T, \qquad (3-2)$$

where $T(a[ij]) = T_k(a)[ij]$, and $\mathbf{T} = (T_1, T_2, T_3)$ is as (2–2).
In fact, for $D(a[ij], b[ij])$, we have

$$D(a[ij], b[ij])d = \gamma_i \gamma_j^{-1}(T_k(d), T_i(d), T_j(d)).$$

Thereby we have the following.

Theorem 3.1. *Let A, B be as defined above. If we assume that the identities (2–3) and the following*

$$\bar{c}((\bar{a}b)d) - (ab)(cd) + (d(\bar{b}c))\bar{a} - (da)(bc)$$

$$+d((ca)b) - d(\bar{b}(\bar{c}a)) + b((ca)d) - (\bar{c}a)(\bar{b}d) = 0, \qquad (3-3)$$

then $(B, [\ ,\ ])$ is a generalized structurable algebra equipped with the derivation defined by the identity (3–2).

Proof. First, we must show that

$$D(a[ij], b[ij])[c[ij], d[jk]]$$

$$= [D(a[ij], b[ij])(c[ij]), d[jk]]$$

$$+[c[ij], D(a[ij], b[ij])(d[jk])]$$

for cyclic permutations (i,j,k).
The left hand side is given by

$$\gamma_i \gamma_j^{-1} T(cd[ik])$$

$$= -\gamma_i \gamma_j^{-1}(\gamma_i \gamma_k^{-1}) T_j(\overline{(cd)})[ki].$$

The right hand side is the following:

$$X = a[ij], Y = b[jk], Z = c[ki].$$

This completes the proof.

Hereafter we say that $D(a[ij], b[ij])$ is an inner derivation of B.

Next we shall consider a vector space $K(B) = InnDer\ B \oplus B$ by linearly extending the product on $InnDer\ B$ by defining as follows:

$$[a[ij], b[jk]]^* = -[b[jk], a[ij]]^* = ab[ik] + D(a[ij], b[jk]),$$

$$[T, a[ij]]^* = -[a[ij], T]^* = T_k(a)[ij],$$

$$[a[ij], b[ij]]^* = \gamma_i \gamma_j^{-1} T, \qquad (3-4)$$

where T is as in (3–2), and so $D(a[ij], b[jk]) = 0$ for distinct i, j, k. Then we have the following.

Theorem 3.2. *Let B be as in Theorem 3.1. If we assume the identities (2–3) and (3–3), then the vector space $K(B) = Inn\ Der\ B \oplus B$ is a generalized structurable algebra equipped with new a product and new derivation defined by*

$$[X, Y]^* := D_1 x_2 - D_2 x_1 + [x_1, x_2] + [D_1, D_2] + D(x_1, x_2)$$

$$D^*(X, Y)Z := [[D_1, D_2], D_3] + [D(x_1, x_2), D_3]$$
$$+ D(x_1, x_2)x_3 + [D_1, D_2]x_3,$$

for $X = D_1 + x_1, Y = D_2 + x_2, Z = D_3 + x_3$,
$D_i \in InnDer\ B, x_i \in B$.

Furthermore $K(B)$ becomes a Lie algebra, i.e., $K(B)$ is the standard embedding Lie algebra associated with B.

Proof. From the definition of a Lie related triple, it follows that

$$D^*(X, Y)[Z, W]^* = [D^*(X, Y)Z, W]^* + [Z, D^*(X, Y)W]^*.$$

From the identity (3–3), it follows that

$$D^*([X, Y]^*, Z) + D^*([Y, Z]^*, X) + D^*([Z, X]^*, Y) = 0.$$

Therefore we obtain the result that the vector space

$$K(B) = InnDer\ B \oplus B$$

is a generalized structurable algebra equipped with the dervation $D^*(X,Y)$. Furthermore, we can show that $K(B)$ is a Lie algebra. In fact, by means of the identity (2–3), we obtain

$$[D(X,Y), D(U,V)] = D(D(X,Y)U,V) + D(U, D(X,Y)).$$

On the other hand, for the case of $X = a[ij], Y = b[ij]$, $Z = c[kj]$ with distinct i, j, k, we have

$$\begin{aligned} D(X,Y)Z &= D(a[ij], b[ij])c[kj] \\ &= \gamma_i \gamma_j^{-1} T_i(c[kj]) \\ &= -\gamma_i \gamma_j^{-1} T_i(\gamma_k \gamma_j^{-1} \bar{c}[jk]) \\ &= -\gamma_i \gamma_j^{-1} \gamma_k \gamma_j^{-1} T_i(\bar{c})[jk]. \end{aligned}$$

Similarly, we have

$$\begin{aligned} [[Y,Z],X] &= [b[ij], c[kj]], a[ij]] \\ &= -\gamma_k \gamma_j^{-1} \gamma_i \gamma_j^{-1} \bar{a}(b\bar{c})[jk], \\ [[Z,X],Y] &= \gamma_i \gamma_j^{-1} \gamma_k \gamma_j^{-1} \bar{b}(a\bar{c})[jk], \end{aligned}$$

and from the definition, it follows that

$$D(Y,Z)X = [[X,Y],Z] = D(Z,X)Y = 0.$$

Hence from the identity (2–3), we obtain

$$\sum_{(X,Y,Z)} D(X,Y)Z + [[X,Y],Z] = 0.$$

Similarly, it holds for the other cases. This completes the proof.

Remark. We note that the identity for Lie related triples is equivalent to the identity for derivations $D(X,Y)$.