

**Opera suspicionată (OS)****Opera autentică (OA)****Suspicious work****Authentic work**

OS	Ciobanu, C., "Something about Frattini subalgebras of a class of solvable Lie algebras", <i>Analele St. Univ.Ovidius Constanța</i> , vol.9 nr.2, 2001, p.9-16.
OA	Stitzinger, E.L., "Frattini subalgebras of a class of solvable Lie Algebras", <i>Pacific Journal of mathematics</i> , vol.34, No.1, 1970, p.177-182.

**Incidența minimă a suspiciunii / Minimum incidence of suspicion**

p.9:20 –p.12:12

p.177:15 – p.179:3

p.12:18 – p.14:15

p.179:34 - p.181:13

Fișa întocmită pentru includerea suspiciunii în Indexul Operelor Plagiate în România de la  
[www.plagiate.ro](http://www.plagiate.ro)

## FRATTINI SUBALGEBRAS OF A CLASS OF SOLVABLE LIE ALGEBRAS

ERNEST L. STITZINGER

In this paper the Lie algebra analogues to groups with property  $E$  of Bechtell are investigated. Let  $\mathfrak{X}$  be the class of solvable Lie algebras with the following property: if  $H$  is a subalgebra of  $L$ , then  $\phi(H) \subseteq \phi(L)$  where  $\phi(L)$  denotes the Frattini subalgebra of  $L$ ; that is,  $\phi(L)$  is the intersection of all maximal subalgebras of  $L$ . Groups with the analogous property are called  $E$ -groups by Bechtell. The class  $\mathfrak{X}$  is shown to contain all solvable Lie algebras whose derived algebra is nilpotent. Necessary conditions are found such that an ideal  $N$  of  $L \in \mathfrak{X}$  be the Frattini subalgebra of  $L$ . Only solvable Lie algebras of finite dimension are considered here.

The following notation will be used. We let  $N(L)$  be the nil radical of  $L$  and  $S(L)$  be the socle of  $L$ ; that is,  $S(L)$  is the union of all minimal ideals of  $L$ . If  $A$  and  $B$  are subalgebras of  $L$ , let  $Z_B(A)$  be the centralizer of  $A$  in  $B$ . The center of  $A$  will be denoted by  $Z(A)$ . If  $[B, A] \subseteq A$ , we let  $\text{Ad}_A(B) = \{\text{ad } b \text{ restricted to } A; \text{ for all } b \in B\}$ .  $L'$  will be the derived algebra of  $L$  and  $L'' = (L)'$ .

PROPOSITION 1. *Let  $L$  be a Lie algebra such that  $L'$  is nilpotent. Then the following are equivalent:*

- (1)  $\phi(L) = 0$ .
- (2)  $N(L) = S(L)$  and  $N(L)$  is complemented by a subalgebra.
- (3)  $L'$  is abelian, is a semi-simple  $L$ -module and is complemented by a subalgebra.

*Under these conditions, Cartan subalgebras of  $L$  are exactly those subalgebras complementary to  $L'$ .*

*Proof.* Assume (1) holds. Nilpotency of  $L'$  implies  $\phi(L) \supseteq L''$ , so  $L'$  is abelian and may be regarded as an  $L/L'$ -module. We may assume  $L' = \sum \bigoplus V_\rho$ ,  $V_\rho$  indecomposable  $L/L'$ -submodules. If  $M$  is a maximal subalgebra of  $L$  and if  $V_\rho \not\subseteq M$ , then  $M \cap V_\rho$  is an ideal of  $L$ . If  $S$  is an  $L/L'$ -submodule of  $V_\rho$  properly contained between  $M \cap V_\rho$  and  $V_\rho$ , then  $M + S$  is a subalgebra of  $L$  properly contained between  $M$  and  $L$ , contradicting the maximality of  $M$ . Therefore  $M$  contains all maximal submodules of  $V_\rho$  for each  $\rho$ . Then  $\phi(L) = 0$  implies the intersection of all maximal submodules of  $V_\rho$  is zero for each  $\rho$ . If  $V_1, \dots, V_s$  are maximal submodules of  $V_\rho$  with  $V_1 \cap \dots \cap V_s = 0$  and are minimal with respect to this property, we have  $V = V_2 \cap \dots \cap V_s \neq 0$

and  $V \cap V_1 = 0$  so that  $V \oplus V_1 = V_\rho$ , contradicting indecomposability. Therefore each  $V_\rho$  is irreducible and  $L'$  is a completely reducible  $L/L'$ -module and is also a completely reducible  $L$ -module. Since  $L$  is solvable it contains Cartan subalgebras by Theorem 3 of [1]. Let  $H$  be a Cartan subalgebra of  $L$  and let  $L_0$  and  $L_1$  be the Fitting null and one component of  $L$  with respect to  $H$ . Then  $L = L_0 + L_1 = H + L_1 \cong H + L'$  shows  $L = H + L'$ . We claim that  $H \cap L' = 0$ . If  $H \cap L' \neq 0$ , then, since  $L'$  is abelian,  $H$  is nilpotent and  $L'$  is a completely reducible  $L$ -module,  $L'$  is a sum of irreducible  $H$ -modules,  $U_1, \dots, U_r$ , such that for each  $U_i$   $[\dots \underbrace{[U_i, H] \dots}_k H] = 0$  for some  $k$ , hence  $[U_i, H] = 0$ .

Thus  $[H, L' \cap H] = 0$ . One sees that each  $U_i$  is a central minimal ideal of  $L$ , and since  $\phi(L) = 0$ ,  $U_i$  is complemented by a maximal subalgebra  $M$ . Therefore  $U_i$  is a one-dimensional direct summand of  $L$ , contradicting  $U_i \subseteq L'$ . Hence  $L' \cap H = 0$  and  $H$  is a complement to  $L'$  in  $L$ . Since  $[H, H] \subseteq H \cap L' = 0$ ,  $H$  is abelian. Any minimal ideal not in  $L'$  satisfies  $[L, A] \subseteq A \cap L' = 0$ , so is central. Therefore  $S(L) = L' + Z(L)$  and, since  $H$  is a Cartan subalgebra,  $Z(L) \subseteq H$ . Let  $H_0$  be a complementary subspace to  $Z(L)$  in  $H$ . One sees that  $N(L) = L' + Z(L) + (N(L) \cap H_0) = S(L) + (N(L) \cap H_0)$ . If  $h$  is a nonzero element in  $N(L) \cap H_0$ ,  $\text{ad } h$  is nilpotent but not zero which implies  $[V_\rho, h] = V_\rho$  for some  $V_\rho \subseteq L'$  and  $[\dots \underbrace{[V_\rho, h] \dots}_k h] = 0$  for some  $k$ , a contradiction. Thus  $S(L) = N(L)$  and  $H_0$  is a complement. Consequently (1) implies (2).

Assume (2) holds and proceed by induction on the dimension of  $L$ . Since  $L' \subseteq N(L) = S(L)$  and minimal ideals are abelian,  $L'$  is abelian. If every minimal ideal of  $L$  is contained in  $L'$ , then  $S(L) = L'$  and (3) follows. Therefore let  $A$  be a minimal ideal of  $L$  such that  $A \not\subseteq L'$ . Hence  $A \not\subseteq \phi(L)$  and there exists a maximal subalgebra  $M$  of  $L$  such that  $L = M + A$ . Since  $[L, A] \subseteq A \cap L' = 0$ ,  $A$  is central, hence one-dimensional. It follows that  $L$  is the Lie algebra direct sum of  $M$  and  $A$ . Since  $M$  inherits the condition (2),  $M$  satisfies (3) by induction. It now follows that  $L$  also satisfies (3).

Assume (3) holds. Then  $L'$  is a sum of minimal ideals of  $L$ , which we denote by  $A_1, \dots, A_k$ , and  $L = L' + H$ ,  $H$  a subalgebra of  $L$ . Since  $H' \subseteq H \cap L' = 0$ ,  $H$  is abelian. One sees that  $L' = [L', H]$  and, consequently,  $A_i = [A_i, H]$  for all  $i$ . Since  $Z_{A_i}(H)$  is central in  $L$ ,  $Z_{A_i}(H)$  is an ideal in  $L$  contained in  $A_i$ . Since  $Z_{A_i}(H) \neq A_i$ ,  $Z_{A_i}(H) = 0$ . It follows that  $H$  is its own normalizer, hence is a Cartan subalgebra of  $L$ . Now  $H + A_1 + \dots + \hat{A}_i + \dots + A_k$  is a maximal subalgebra of  $L$  since any containing algebra has a nonzero projection on  $A_i$  which is  $\text{ad } H$  stable, hence equal to  $A_i$ . Therefore  $\phi(L) \subseteq H$  and  $\phi(L) \subseteq H \cap L' = 0$ . Hence (1) holds.

That complements to  $L'$  are Cartan subalgebras is shown in (3) implies (1). That Cartan subalgebras are complements to  $L'$  is shown in (1) implies (2). This completes the proof of Proposition 1.

**THEOREM 1.** *Let  $L$  be a Lie algebra such that  $L'$  is nilpotent and  $\phi(L) = 0$ . Then, for any subalgebra  $M$  of  $L$ ,  $\phi(M) = 0$ .*

*Proof.* Suppose  $L' \subseteq M$ . Let  $H$  be a complement to  $L'$  in  $L$ , so  $H \cap M$  is a complement to  $L'$  in  $M$ . Since  $L$  acts completely reducibly on  $L'$  and  $L'$  is abelian,  $H$  acts completely reducibly on  $L'$ . Then, since  $H$  is abelian,  $H \cap M$  acts completely reducibly on  $L'$ , hence so does  $M$ . Therefore  $L' = M' \oplus A$  for some ideal  $A$  in  $M$  where  $M$  acts completely reducibly on  $M'$  and  $A + (H \cap M)$  is a complementary subalgebra of  $M'$  in  $M$ . Thus by Proposition 1,  $\phi(M) = 0$ .

Suppose  $L' \not\subseteq M$ . Since  $M + L'$  falls in the preceding case, we may assume  $M + L' = L$ . Since  $L'$  is abelian,  $L' \cap M$  is an ideal in  $L$ ,  $M/(L' \cap M)$  complements  $L'/(L' \cap M) = (L/L' \cap M)'$  in  $L/(L' \cap M)$  and  $M/(L' \cap M)$  acts completely reducibly in  $L'/(L' \cap M)$ ,  $M/(L' \cap M)$  is a Cartan subalgebra of  $L/(L' \cap M)$ . Let  $C$  be a Cartan subalgebra of  $M$ . By Lemma 4 of [1],  $C$  is a Cartan subalgebra of  $L$ . Thus  $C$  is a complement to  $L'$  and  $C + (L' \cap M) = M$  since  $C \subseteq M$ . Hence  $C$  is a complement to  $L' \cap M$  in  $M$ . Since  $M$  acts completely reducibly on  $L' \cap M$  and  $M' \subseteq L' \cap M$ ,  $M$  acts completely reducibly on  $M'$ ,  $L' \cap M = M' \oplus (L' \cap Z(M))$  and, since  $Z(M) \subseteq C$ ,  $Z(M) \cap L' \subseteq C \cap L' = 0$ . Therefore  $C = M' = M$  and  $C \cap M' = 0$ . Now  $M$  satisfies part (3) of Proposition 1, hence  $\phi(M) = 0$ .

If  $L$  is a solvable Lie algebra it has been shown in [2] that  $\phi(L)$  is an ideal of  $L$ . We look for a condition on the subalgebras of  $L/\phi(L)$  which are necessary and sufficient that  $L \in \mathfrak{X}$ . In order to do this the following concept is introduced.

We shall say that a Lie algebra  $L$  is the reduced partial sum of an ideal  $A$  and a subalgebra  $B$  if  $L = A + B$  and for any subalgebra  $C$  of  $L$  such that  $L = A + C$  and  $C \subseteq B$  then  $C = B$ . It is noted that if  $A \not\subseteq \phi(L)$ , then there exists a  $B \neq L$  such that  $L$  is the reduced partial sum of  $A$  and  $B$ . On the other hand, if  $A \subseteq \phi(L)$  and  $L$  is the reduced partial sum of  $A$  and  $B$ , then  $B = L$ .

**LEMMA 1.** *Let  $L$  be the reduced partial sum of  $A$  and  $B$ . Then  $A \cap B \subseteq \phi(B)$ .*

*Proof.* Suppose  $C = A \cap B \not\subseteq \phi(B)$ . Then  $B$  contains a subalgebra  $D$  such that  $C + D = B$ . Then  $L = A + B = A + C + D = A + D$ . This contradicts the minimality of  $B$ .

**LEMMA 2.** *Let  $L$  be the reduced partial sum of  $A$  and  $B$ . Then  $\phi(L/A) \simeq A + \phi(B)/A$ .*

*Proof.* Since  $A \cap B \subseteq \phi(B)$ ,  $A \cap \phi(B) = A \cap B$ . Since  $L/A \simeq A + B/A \simeq B/A \cap B$ ,  $\phi(L/A) \simeq \phi(B/A \cap B) \simeq \phi(B)/A \cap B = \phi(B)/A \cap \phi(B) \simeq A + \phi(B)/A$ .

**PROPOSITION 2.** *The following are equivalent for the Lie algebra  $L$ :*

- (1)  $L \in \mathfrak{X}$ .
- (2) For any subalgebra  $H$  of  $L/\phi(L)$ ,  $\phi(H) = 0$ .

*Proof.* Let  $L$  satisfy (1) and let  $\pi: L \rightarrow L/\phi(L)$  be the natural homomorphism. Then  $\phi(\pi(L)) = \pi(\phi(L)) = 0$ . Let  $\bar{W}$  be a subalgebra of  $L/\phi(L)$  and let  $W$  be the subalgebra of  $L$  which contains  $\phi(L)$  and corresponds to  $\bar{W}$ . Since  $L$  satisfies (1),  $\phi(W) \subseteq \phi(L)$ . If  $\phi(W) = \phi(L)$ , then  $\phi(\pi(W)) = \pi(\phi(W)) = \pi(\phi(L)) = 0$ . Suppose then that  $\phi(W) \subset \phi(L)$ . Then  $W$  can be represented as a reduced partial sum  $W = \phi(L) + K$ . Let  $T$  be a subalgebra of  $W$  such that  $T/\phi(L) \simeq \phi(W/\phi(L))$ . If  $T/\phi(L) \neq 0$ , then  $T = T \cap (\phi(L) + K) = (T \cap \phi(L)) + (T \cap K) = \phi(L) + (T \cap K)$ . Consequently there exists an  $x \in T \cap K$ ,  $x \notin \phi(L)$ . Since  $\phi(K) \subseteq \phi(L)$ ,  $x \notin \phi(K)$  and there exists a maximum subalgebra  $S$  of  $K$  such that  $x \in S$ . We claim that either  $\phi(L) + S = W$  or  $\phi(L) + S$  is maximal in  $W$ . Suppose  $\phi(L) + S \neq W$  and let  $J$  be a subalgebra of  $W$  which contains  $\phi(L) + S$ . Then  $S \subseteq J \cap K$ , so, by the maximality of  $S$ , either  $J \cap K = S$  or  $J \cap K = K$ . If  $J \cap K = S$ , then  $\phi(L) + S = \phi(L) + (J \cap K) = J \cap (\phi(L) + K) = J \cap W = J$ . If  $J \cap K = K$ , then  $J \supseteq K$  and, since  $J \supseteq \phi(L)$ ,  $J \supseteq \phi(L) + K = W$ , hence  $J = W$ . Consequently there exist no subalgebras of  $W$  properly contained between  $\phi(L) + S$  and  $W$ , hence either  $\phi(L) + S = W$  or  $\phi(L) + S$  is maximal in  $W$ . If  $\phi(L) + S = W$ , then  $\phi(L) + K$  is not a reduced partial sum which is a contradiction. If  $\phi(L) + S$  is maximal in  $W$ , then  $\phi(L) + S/\phi(L) \supseteq \phi(W/\phi(L)) \simeq T/\phi(L)$ . Hence  $T \subseteq \phi(L) + S$ . Since  $S \subseteq \phi(L) + S$  and  $x \in T \cap K \subset T \subseteq \phi(L) + S$ ,  $K = \{S, x\} \subseteq \phi(L) + S$ . Then  $W = \phi(L) + K \subseteq \phi(L) + S \subseteq W$  implies  $\phi(L) + K$  is not a reduced partial sum, a contradiction. Hence  $\phi(\bar{W}) = T/\phi(L) = 0$  and (2) is satisfied.

If  $L/\phi(L)$  satisfies (2), then  $\pi(\phi(H)) \subseteq \phi(\pi(H)) = 0$  for every subalgebra  $H$  of  $L$ . Then  $\phi(H) \subseteq \phi(L)$  for every subalgebra  $H$  of  $L$ .

Combining Proposition 2 and Theorem 1 we have

**THEOREM 2.** *Let  $L$  be a Lie algebra such that  $L'$  is nilpotent. Then  $L \in \mathfrak{X}$ .*

**THEOREM 3.** *Let  $L \in \mathfrak{X}$  and let  $T$  be a Lie homomorphism of  $L$ .*

Then  $T(\phi(L)) = \phi(T(L))$ .

*Proof.*  $T(\phi(L))$  is always contained in  $\phi(T(L))$  by Proposition 1 in [6]. If  $N = \text{kernel } T \subseteq \phi(L)$ , then equality holds by Proposition 2 in [6]. Suppose  $N \not\subseteq \phi(L)$ . Let  $L = N + K$  be a reduced partial sum. Using Lemma 2,  $\phi(T(L)) = \phi(L/N) \simeq N + \phi(K)/N = T(\phi(K))$ . Since  $T(N + \phi(L)) = T(\phi(L)) \subseteq \phi(T(L)) = \phi(L/N) = T(N + \phi(K))$ ,  $N + \phi(L) \subseteq N + \phi(K) \subseteq N + \phi(L)$ . Hence  $N + \phi(L) = N + \phi(K)$  and  $\phi(T(L)) = T(\phi(K)) = T(\phi(L))$ .

**THEOREM 4.** *Let  $L \in \mathfrak{X}$ . Necessary conditions that an ideal  $N$  of  $L$  be the Frattini subalgebra of  $L$  are that*

- (1)  $\phi(\text{Ad}_N(L)) = \text{Ad}_N(\phi(L))$ .
- (2) *There exists a subalgebra  $M$  of  $L$  such that  $M/N \simeq \text{Ad}_N(L)/\text{Ad}_N(\phi(L))$ .*

*Proof.* (1) Let  $T$  be the mapping from  $L$  into the derivation algebra of  $N$  by  $T(x) = \text{ad } x$  restricted to  $N$  for all  $x \in L$ . Then  $T(\phi(L)) = \text{Ad}_N(\phi(L)) = \phi(T(L)) = \phi(\text{Ad}_N(L))$ .

(2) Let  $M = Z_L(\phi(L))$ . Suppose that  $M \not\subseteq \phi(L)$  and let  $F = L/\phi(L)$  and  $A = (M + \phi(L))/\phi(L)$ . Since  $\text{Ad}_{\phi(L)}(L) \simeq L/M$  and  $\text{Ad}_{\phi(L)}(\phi(L)) \simeq \phi(L)/Z(\phi(L)) = \phi(L)/M \cap \phi(L) = (M + \phi(L))/M$ ,  $F/A \simeq (L/\phi(L))/(M + \phi(L)/\phi(L)) \simeq L/(M + \phi(L)) \simeq (L/M)/((M + \phi(L))/M) \simeq \text{Ad}_{\phi(L)}(L)/\text{Ad}_{\phi(L)}(\phi(L))$ . Since  $\phi(F) = 0$ , there exists a subalgebra  $D$  in  $F$  such that  $F$  is the reduced partial sum of  $A$  and  $D$ . Using Proposition 2 and Lemma 1,  $A \cap D \subseteq \phi(D) = 0$ , hence  $A \cap D = 0$ . Let  $E$  be the subalgebra of  $L$  which contains  $\phi(L)$  and corresponds to  $D$ . Then  $E/\phi(L) \simeq D \simeq F/A \simeq \text{Ad}_{\phi(L)}(L)/\text{Ad}_{\phi(L)}(\phi(L))$ . If  $M \subseteq \phi(L)$ , then  $\text{Ad}_{\phi(L)}(L)/\text{Ad}_{\phi(L)}(\phi(L)) \simeq (L/M)/(\phi(L)/Z(\phi(L))) = (L/M)/(\phi(L)/M \cap \phi(L)) = (L/M)/(\phi(L)/M) \simeq L/\phi(L)$ .

Related to part (1) of Theorem 4 are the following results.

**THEOREM 5.** *Let  $L \in \mathfrak{X}$  and let  $K$  be an ideal of  $L$  containing  $\phi(L)$ . Then  $\phi(\text{Ad}_K(L)) \simeq \text{Ad}_K(K)$  if and only if  $K = \phi(L) + Z(K)$ .*

*Proof.* Let  $T$  be the Lie homomorphism from  $L$  into the derivation algebra of  $K$  given by  $T(x) = \text{ad } x$  restricted to  $K$  for each  $x \in L$ . Then  $\phi(\text{Ad}_K(L)) = \phi(T(L)) = T(\phi(L)) = \text{Ad}_K(\phi(L)) \simeq \phi(L)/Z_{\phi(L)}(K) = \phi(L)/(Z(K) \cap \phi(L)) \simeq (\phi(L) + Z(K))/Z(K)$ . If  $\phi(L) + Z(K) = K$ , then  $\text{Ad}_K(K) \simeq K/Z(K) = (\phi(L) + Z(K))/Z(K) \simeq \phi(\text{Ad}_K(L))$ . If  $\phi(L) + Z(K) \subset K$ , then  $\text{Ad}_K(K) \simeq K/Z(K) \supset (\phi(L) + Z(K))/Z(K) = \phi(\text{Ad}_K(L))$ .

**THEOREM 6.** *Let  $L \in \mathfrak{X}$  and let  $A$  be an ideal of  $L$  contained in  $\phi(L)$ . Then  $\phi(\text{Ad}_A(L)) \simeq \text{Ad}_A(A)$  if and only if  $\phi(L) = A + Z_{\phi(L)}(A)$ .*

*Proof.* If  $\phi(L) = A + Z_{\phi(L)}(A)$ , then  $\text{Ad}_A(A) = \text{Ad}_A(\phi(L)) = T(\phi(L)) = \phi(T(L)) = \phi(\text{Ad}_A(L))$ .

Conversely,  $\text{Ad}_A(L) \simeq L/Z_L(A)$  and  $Z_L(A) + A/Z_L(A) \simeq A/Z(A) = \text{Ad}_A(A)$ . Then  $L/Z_L(A) + A \simeq \text{Ad}_A(L)/\text{Ad}_A(A)$  and  $\phi(L/Z_L(A) + A) \simeq \phi(\text{Ad}_A(L)/\text{Ad}_A(A)) = \phi(\text{Ad}_A(L))/\text{Ad}_A(A) = 0$ . Hence  $\phi(L) \subseteq Z_L(A) + A$  and  $\phi(L) = Z_{\phi(L)}(A) + A$ .

The author wishes to thank the referee for many helpful comments. In particular the present form of Proposition 1 and Theorem 1 are his generalizations to results originally submitted.

#### REFERENCES

1. D. W. Barnes, *On Cartan subalgebras of Lie algebras*, Math Z. **101** (1967) 350-355.
2. D. W. Barnes, and H. M. Gastineau-Hills, *On the theory of solvable Lie algebras*, Math Z. **106** (1968) 343-354.
3. H. Bechtell, *Frattini subgroups and  $\phi$ -central groups*, Pacific J. Math **18** (1966), 15-23.
4. ———, *Elementary groups*, Trans. Amer. Math. Soc. **114** (1965), 355-362.
5. N. Jacobson, *Lie algebras*, Interscience.
6. E. Stitzinger, *On the Frattini subalgebra of a Lie algebra*, J. Lond. Math. Soc. (to appear)

Received October 8, 1969.

NORTH CAROLINA STATE UNIVERSITY  
RALEIGH, NORTH CAROLINA