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SOMETHING ABOUT FRATTINI SUBALGEBRAS OF A CLASS OF SOLVABLE LIE ALGEBRAS

Camelia Ciobanu

Dedicated to Professor Mirela Ștefănescu on the occasion of her 60th birthday

Abstract

In this paper we consider \mathcal{Q} the class of solvable Lie algebras \mathcal{L} with the following property: if \mathcal{A} is a subalgebra of \mathcal{L} , then $\Phi(\mathcal{A}) \subseteq \Phi(\mathcal{L})$ (where $\Phi(\mathcal{L})$ denotes the Frattini subalgebra of \mathcal{L} ; that is $\Phi(\mathcal{L})$ is the intersection of all maximal subalgebras of \mathcal{L}). The class \mathcal{Q} is shown to contain all solvable Lie algebras whose derived algebra is nilpotent. Necessary conditions are found such that an ideal \mathcal{I} of $\mathcal{L} \in \mathcal{Q}$ be the Frattini subalgebra of \mathcal{L} . We considered here only solvable Lie algebras of finite dimension.

1. Introduction

In this section, we recall some notions and properties necessary in the paper.

We shall define by $\Phi(\mathcal{L})$ the Frattini subalgebra of \mathcal{L} , to be the intersection of all maximal subalgebras of \mathcal{L} .

We let $\mathcal{N}(\mathcal{L})$ be the nilradical of \mathcal{L} and $\mathcal{S}(\mathcal{L})$ be the socle of \mathcal{L} ; that is $\mathcal{S}(\mathcal{L})$ is the union of all minimal ideals of \mathcal{L} . If \mathcal{A} and \mathcal{B} are subalgebras of \mathcal{L} , let $Z_{\mathcal{B}}(\mathcal{A})$ be the centralizer of \mathcal{A} in \mathcal{B} .

The center of \mathcal{A} will be denoted by $Z(\mathcal{A})$. If $[\mathcal{B}, \mathcal{A}] \subseteq \mathcal{A}$, we let

$$Ad_{\mathcal{A}}(\mathcal{B}) = \{adb \text{ restricted to } \mathcal{A}; \text{ for all } b \in \mathcal{B}\}.$$

\mathcal{L}' will be the derived algebra of \mathcal{L} and $\mathcal{L}'' = (\mathcal{L}')'$.

Key Words: p -Lie algebra, p -ideal, p -subalgebra, Cartan subalgebra

2.Lie algebras \mathcal{L} with \mathcal{L}' nilpotent

Proposition 2.1. *Let \mathcal{L} be a Lie algebra such that \mathcal{L}' is nilpotent. Then the following are equivalent.*

- (i) $\Phi(\mathcal{L}) = 0$.
- (ii) $\mathcal{N}(\mathcal{L}) = \mathcal{S}(\mathcal{L})$ and $\mathcal{N}(\mathcal{L})$ is complemented by a subalgebra.
- (iii) \mathcal{L}' is abelian, is a semi-simple \mathcal{L} -module and is complemented by a subalgebra.

Under these conditions, Cartan subalgebras of \mathcal{L} are exactly those subalgebras complementary to \mathcal{L}' .

Proof. Assume (i) holds. Nilpotency of \mathcal{L}' implies $\Phi(\mathcal{L}) \supseteq \mathcal{L}'$, so \mathcal{L}' is abelian and may be regarded as an \mathcal{L}/\mathcal{L}' -module. We may assume $\mathcal{L}' = \sum \oplus \mathcal{V}_\rho$, \mathcal{V}_ρ indecomposable \mathcal{L}/\mathcal{L}' -submodules. If \mathcal{M} is a maximal subalgebra of \mathcal{L} and if $\mathcal{V}_\rho \not\subseteq \mathcal{M}$, then $\mathcal{M} \cap \mathcal{V}_\rho$ is an ideal of \mathcal{L} . If \mathcal{J} is an \mathcal{L}/\mathcal{L}' -submodule of \mathcal{V}_ρ properly contained between $\mathcal{M} \cap \mathcal{V}_\rho$ and \mathcal{V}_ρ , then $\mathcal{M} + \mathcal{J}$ is a subalgebra of \mathcal{L} properly contained between \mathcal{M} and \mathcal{L} , contradicting the maximality of \mathcal{M} . Therefore \mathcal{M} contains all maximal submodules of \mathcal{V}_ρ for each ρ . Then $\Phi(\mathcal{L}) = 0$ implies the interections of all maximal submodules of \mathcal{V}_ρ is zero for each ρ . If V_1, V_2, \dots, V_s are maximal submodule of \mathcal{V}_ρ with $V_1 \cap \dots \cap V_s = 0$ and are minimal with respect to this property, we have $V = V_2 \cap \dots \cap V_s \neq 0$ and $V \cap V_1 = 0$ so that $V \oplus V_1 = \mathcal{V}_\rho$, contradicting indecomposability. Therefore each \mathcal{V}_ρ is irreducible and \mathcal{L}' is a completely reducible \mathcal{L}/\mathcal{L}' -module and is also a completely reducible \mathcal{L} -module. Since \mathcal{L} is solvable it contains Cartan subalgebras by Theorem 3 of [1]. Let \mathcal{C} be a Cartan subalgebra of \mathcal{L} and let \mathcal{L}_0 and \mathcal{L}_1 , be the Fitting null and one component of \mathcal{L} with respect to \mathcal{C} . Then $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 \equiv \mathcal{C} + \mathcal{L}_1 \subseteq \mathcal{C} + \mathcal{L}'$ shows $\mathcal{L} = \mathcal{C} + \mathcal{L}'$. We claim that $\mathcal{C} \cap \mathcal{L}' = 0$. If $\mathcal{C} \cap \mathcal{L}' \neq 0$, then, since \mathcal{L}' is abelian, \mathcal{C} is nilpotent and \mathcal{L}' is a completely reducible \mathcal{L} -module, \mathcal{L}' is a sum of irreducible \mathcal{C} -modules, $\mathcal{U}_1, \dots, \mathcal{U}_q$, such that for each \mathcal{U}_i $\underbrace{[\dots, [\mathcal{U}_i, \mathcal{C}], \dots \mathcal{C}]}_k = 0$ for some k , hence $[\mathcal{U}_i, \mathcal{C}] = 0$. Thus $[\mathcal{C}$,

$\mathcal{L}' \cap \mathcal{C}] = 0$. One sees that each \mathcal{U}_i is a central minimal ideal of \mathcal{L} , and since $\Phi(\mathcal{L}) = 0$, \mathcal{U}_i is complemented by a maximal subalgebra \mathcal{M} . Therefore \mathcal{U}_i is a one-dimensional direct summand of \mathcal{L} , contradicting $\mathcal{U}_i \subseteq \mathcal{L}'$. Hence $\mathcal{L}' \cap \mathcal{C} = 0$ and \mathcal{C} is a complement to \mathcal{L}' in \mathcal{L} . Since $[\mathcal{C}, \mathcal{C}] \subseteq \mathcal{C} \cap \mathcal{L}' = 0$, \mathcal{C} is abelian. Any minimal ideal not in \mathcal{L}' satisfies $[\mathcal{L}, \mathcal{A}] \subseteq \mathcal{A} \cap \mathcal{L}' = 0$, so is central.

Therefore $\mathcal{S}(\mathcal{L}) = \mathcal{L}' + Z(\mathcal{L})$ and, since \mathcal{C} is a Cartan subalgebra, $Z(\mathcal{L}) \subseteq \mathcal{C}$. Let \mathcal{C}_0 be a complementary subspace to $Z(\mathcal{L})$ in \mathcal{C} .

One sees that

$$\mathcal{N}(\mathcal{L}) = \mathcal{L}' + Z(\mathcal{L}) + (\mathcal{N}(\mathcal{L}) \cap \mathcal{C}_0) = \mathcal{S}(\mathcal{L}) + (\mathcal{N}(\mathcal{L}) \cap \mathcal{C}_0).$$

If c is a nonzero element in $\mathcal{N}(\mathcal{L}) \cap \mathcal{C}_0$, and c is nilpotent but not zero with implies $[V_\rho, c] = V_\rho$ for some $V_\rho \subseteq \mathcal{L}'$ and $\underbrace{[\dots[V_\rho, c]\dots c]}_k = 0$ for some k , a contradicting. This $\mathcal{S}(\mathcal{L}) = \mathcal{N}(\mathcal{L})$ and \mathcal{C}_0 is a complement. Consequently (i) implies (ii).

Assume (ii) holds and proceed by induction on the dimension of \mathcal{L} . Since $\mathcal{L}' \subseteq \mathcal{N}(\mathcal{L}) = \mathcal{S}(\mathcal{L})$ and minimal ideals are abelian, \mathcal{L}' is abelian. If every minimal ideal of \mathcal{L} is contained in \mathcal{L}' , then $\mathcal{S}(\mathcal{L}) = \mathcal{L}'$ and (iii) follows. Therefore let \mathcal{J} be the minimal ideal of \mathcal{L} such that $\mathcal{J} \not\subseteq \mathcal{L}'$.

Hence $\mathcal{J} \not\subseteq \Phi(\mathcal{L})$ and there exists a maximal subalgebra \mathcal{M} of \mathcal{L} such that $\mathcal{L} = \mathcal{M} + \mathcal{J}$.

Since $[\mathcal{L}, \mathcal{J}] \subseteq \mathcal{J} \cap \mathcal{L}'$, \mathcal{J} is central, hence one-dimensional. It follows that \mathcal{L} is the Lie algebra direct sum of \mathcal{M} and \mathcal{J} . Since \mathcal{M} inherits the condition (ii), \mathcal{M} satisfies (iii) by induction. It now follows that \mathcal{L} also satisfies (iii). Assume (iii) holds. Then \mathcal{L}' is a sum of minimal ideals of \mathcal{L} , which we denote by $\mathcal{J}_1, \mathcal{J}_2, \dots, \mathcal{J}_k$ and $\mathcal{L} = \mathcal{L}' + \mathcal{D}$, \mathcal{D} a subalgebra of \mathcal{L} . Since $\mathcal{D}' \subseteq \mathcal{D} \cap \mathcal{L}' = 0$, \mathcal{D} is abelian. One sees that $\mathcal{L}' = [\mathcal{L}', \mathcal{D}]$ and, consequently, $\mathcal{J}_i = [\mathcal{J}_i, \mathcal{D}]$ for all i . Since $Z_{\mathcal{J}}(\mathcal{D})$ is central in \mathcal{L} , $Z_{\mathcal{J}_i}(\mathcal{D})$ is an ideal in \mathcal{L} contained in \mathcal{J}_i . Since $Z_{\mathcal{J}_i}(\mathcal{D}) \neq \mathcal{J}_i$, $Z_{\mathcal{J}_i}(\mathcal{D}) = 0$. It follows that \mathcal{D} is its own normalizer, hence is a Cartan subalgebra of \mathcal{L} .

Now $\mathcal{D} + \mathcal{J}_1 + \dots + \widehat{\mathcal{J}_i} + \dots + \mathcal{J}_k$ is a maximal subalgebra of \mathcal{L} since any containing algebra has a nonzero projection on \mathcal{J}_i which is $ad \mathcal{D}$ stable, hence equal to \mathcal{J}_i . Therefore $\Phi(\mathcal{L}) \subseteq \mathcal{D}$ and $\Phi(\mathcal{L}) \subseteq \mathcal{D} \cap \mathcal{L}' = 0$. Hence (i) holds.

The fact that complements to \mathcal{L}' are Cartan subalgebras is shown in (iii) implies (i). The fact that Cartan subalgebras are complements to \mathcal{L}' is shown in (i) implies (ii). This completes the proof of Proposition 2.1. \square

Theorem 2.1. *Let \mathcal{L} be a Lie algebra such that \mathcal{L}' is nilpotent and $\Phi(\mathcal{L})=0$. Then, for any subalgebra \mathcal{D} of \mathcal{L} , $\Phi(\mathcal{D}) = 0$.*

Proof. Suppose $\mathcal{L}' \subseteq \mathcal{D}$. Let ξ be a complement to \mathcal{L}' in \mathcal{L} , so $\xi \cap \mathcal{D}$ is a complement to \mathcal{L}' in \mathcal{D} . Since \mathcal{L} acts completely reducibly on \mathcal{L}' and \mathcal{L}' is abelian, ξ acts completely reducible on \mathcal{L}' . Then, since ξ is abelian, $\mathcal{D} \cap \xi$ acts completely reducibly on \mathcal{L}' , hence so does \mathcal{D} .

Therefore $\mathcal{L}' = \mathcal{D}' \oplus \mathcal{J}$ for some ideal \mathcal{J} in \mathcal{D} when \mathcal{D} acts completely reducibly on \mathcal{D}' and $\mathcal{J} + (\xi \cap \mathcal{D})$ is a complementary subalgebra of \mathcal{D}' in \mathcal{D} . Thus, by Proposition 2.1, $\Phi(\mathcal{D}) = 0$.

Suppose $\mathcal{L}' \not\subseteq \mathcal{D}$. Since $\mathcal{D} + \mathcal{L}'$ falls in the preceding case, we may assume $\mathcal{D} + \mathcal{L}' = \mathcal{L}$. Since \mathcal{L}' is abelian, $\mathcal{L}' \cap \mathcal{D}$ is an ideal in \mathcal{L} , $\mathcal{D}/(\mathcal{L}' \cap \mathcal{D})$ complements $\mathcal{L}'/(\mathcal{L}' \cap \mathcal{D}) = (\mathcal{L}'/\mathcal{L}' \cap \mathcal{D})$ in $\mathcal{L}/(\mathcal{L}' \cap \mathcal{D})$ and $\mathcal{D}/(\mathcal{L}' \cap \mathcal{D})$ acts completely reducibly in $\mathcal{L}'/(\mathcal{L}' \cap \mathcal{D})$, $\mathcal{D}/(\mathcal{L}' \cap \mathcal{D})$ is a Cartan subalgebra of $\mathcal{L}/(\mathcal{L}' \cap \mathcal{D})$.

Let ζ be a Cartan subalgebra of \mathcal{D} , so it is a Cartan subalgebra of \mathcal{L} . This ζ is a complement to \mathcal{L}' and $\zeta + (\mathcal{L}' \cap \mathcal{D}) = \mathcal{D}$ since $\subseteq \mathcal{D}$. Hence ζ is a complement to $\mathcal{L}' \cap \mathcal{D}$ in \mathcal{D} . Since \mathcal{D} acts completely reducibly on $\mathcal{L}' \cap \mathcal{D}$ and $\mathcal{D}' \subseteq \mathcal{L}' \cap \mathcal{D}$, \mathcal{D} acts completely reducibly on \mathcal{D}' ,

$$\mathcal{L}' \cap \mathcal{D} = \mathcal{D}' \oplus (\mathcal{L}' \cap Z(\mathcal{D}))$$

and, since $Z(\mathcal{D}) \subseteq \zeta$,

$$Z(\mathcal{D}) \cap \mathcal{L}' \subseteq \zeta \cap \mathcal{L} = 0.$$

Therefore $\zeta = \mathcal{D}' = \mathcal{D}$ and $\zeta \cap \mathcal{D}' = 0$. Now \mathcal{D} satisfies part (iii) of Proposition 2.1 hence, $\Phi(\mathcal{D}) = 0$. \square

If \mathcal{L} is a solvable Lie algebra it has been shown that $\Phi(\mathcal{L})$ is an ideal of \mathcal{L} . We look for a condition on the subalgebras of $\mathcal{L}/\Phi(\mathcal{L})$ which are necessary and sufficient that $\mathcal{L} \in \mathcal{Q}$. In order to do this, the following concept is introduced.

Definition 2.1. We shall say that a Lie algebra \mathcal{L} is *the reduced partial sum of an ideal \mathcal{J} and a subalgebra \mathcal{A}* if $\mathcal{L} = \mathcal{A} + \mathcal{J}$ and for any subalgebra \mathcal{D} of \mathcal{L} such that $\mathcal{L} = \mathcal{J} + \mathcal{D}$ and $\mathcal{D} \subseteq \mathcal{A}$ then $\mathcal{D} = \mathcal{A}$.

It is noted that if $\mathcal{J} \not\subseteq \Phi(\mathcal{L})$, then there exists a subalgebra $\mathcal{A} \neq \mathcal{L}$ such that \mathcal{L} is the reduced partial sum of \mathcal{J} and \mathcal{A} . On the other hand, if $\mathcal{J} \subseteq \Phi(\mathcal{L})$ and \mathcal{L} is the reduced partial sum of \mathcal{J} and \mathcal{A} , then $\mathcal{A} = \mathcal{L}$.

Lemma 2.1. *Let \mathcal{L} be the reduced partial sum of \mathcal{J} and \mathcal{A} . Then $\mathcal{J} \cap \mathcal{A} \subseteq \Phi(\mathcal{A})$.*

Proof. Suppose $\mathcal{B} = \mathcal{J} \cap \mathcal{A} \not\subseteq \Phi(\mathcal{A})$. Then \mathcal{A} contains a subalgebra \mathcal{D} such that $\mathcal{B} + \mathcal{D} = \mathcal{A}$. Then $\mathcal{L} = \mathcal{J} + \mathcal{A} = \mathcal{J} + \mathcal{B} + \mathcal{D} = \mathcal{J} + \mathcal{D}$. This contradicts the minimality of \mathcal{A} , then $\mathcal{J} \cap \mathcal{A} \subseteq \Phi(\mathcal{A})$. \square

Lemma 2.2. *Let \mathcal{L} be the reduced partial sum of \mathcal{J} and \mathcal{A} . Then $\Phi(\mathcal{L}/\mathcal{J}) \cong \mathcal{J} + \Phi(\mathcal{A})/\mathcal{J}$.*

Proof. Since $\mathcal{J} \cap \mathcal{A} \subseteq \Phi(\mathcal{A})$, $\mathcal{J} \cap \Phi(\mathcal{A}) = \mathcal{J} \cap \mathcal{A}$. Since

$$\mathcal{L}/\mathcal{J} \cong \mathcal{A} + \mathcal{A}/\mathcal{J} \cong \mathcal{A}/\mathcal{J} \cap \mathcal{A},$$

$\Phi(\mathcal{L}/\mathcal{J}) \cong \Phi(\mathcal{A}/\mathcal{J} \cap \mathcal{A}) \cong \Phi(\mathcal{A})/\mathcal{J} \cap \mathcal{A} = \Phi(\mathcal{A})/\mathcal{J} \cap \Phi(\mathcal{A}) \cong \mathcal{J} + \Phi(\mathcal{A})/\mathcal{J}$. \square

Proposition 2.2. *For a Lie algebra \mathcal{L} , the following statements are equivalent:*

(ii) $\mathcal{L} \in \mathcal{Q}$.

(ii) For any subalgebra \mathcal{A} of $\mathcal{L}/\Phi(\mathcal{L})$, $\Phi(\mathcal{A}) = 0$.

Proof. Let \mathcal{L} satisfy (i) and let $\pi : 1 \rightarrow \mathcal{L}/\Phi(\mathcal{L})$ be the natural homomorphism. Then $\Phi(\pi(\mathcal{L})) = \pi(\Phi(\mathcal{L})) = 0$.

Let $\overline{\mathcal{B}}$ be a subalgebra of $\mathcal{L}/\Phi(\mathcal{L})$ and let \mathcal{B} be the subalgebra of \mathcal{L} which contains $\Phi(\mathcal{L})$ and corresponds to $\overline{\mathcal{B}}$. Since \mathcal{L} satisfies (i), $\Phi(\mathcal{B}) \subseteq \Phi(\mathcal{L})$. If $\Phi(\mathcal{B}) = \Phi(\mathcal{L})$, then

$$\Phi(\pi(\mathcal{B})) = \pi(\Phi(\mathcal{B})) = \pi(\Phi(\mathcal{L})) = 0.$$

Suppose then that $\Phi(\mathcal{B}) \subset \Phi(\mathcal{L})$. Then \mathcal{B} can be represented as a reduced partial sum $\mathcal{B} = \Phi(\mathcal{L}) + \mathcal{P}$. Let \mathcal{R} be a subalgebra of \mathcal{B} such that $\mathcal{R}/\Phi(\mathcal{L}) \cong \Phi(\mathcal{B}/\Phi(\mathcal{L}))$. If $\mathcal{R}/\Phi(\mathcal{L}) \neq 0$, then

$$\mathcal{R} = \mathcal{R} \cap (\Phi(\mathcal{L}) + \mathcal{P}) = (\mathcal{R} \cap \Phi(\mathcal{L})) + (\mathcal{R} \cap \mathcal{P}) = \Phi(\mathcal{L}) + (\mathcal{R} \cap \mathcal{P}).$$

Consequently there exists an $x \in \mathcal{R} \cap \mathcal{P}$, $x \notin \Phi(\mathcal{L})$. Since $\Phi(\mathcal{P}) \subseteq \Phi(\mathcal{L})$, $x \notin \Phi(\mathcal{P})$ and there exists a maximum subalgebra \mathcal{M} of \mathcal{P} such that $x \notin \mathcal{M}$. We claim that either $\Phi(\mathcal{L}) + \mathcal{M} = \mathcal{B}$ or $\Phi(\mathcal{L}) + \mathcal{M}$ is maximal in \mathcal{B} .

Suppose $\Phi(\mathcal{L}) + \mathcal{M} \neq \mathcal{B}$ and let J be a subalgebra of \mathcal{B} which contains $\Phi(\mathcal{L}) + \mathcal{M}$. Then $\mathcal{M} \subseteq J \cap \mathcal{P}$, so, by the maximality of \mathcal{M} , either $J \cap \mathcal{P} = \mathcal{M}$ or $J \cap \mathcal{P} = \mathcal{P}$. If $J \cap \mathcal{P} = \mathcal{M}$ then

$$\Phi(\mathcal{L}) + \mathcal{M} = \Phi(\mathcal{L}) + (J \cap \mathcal{P}) = J \cap (\Phi(\mathcal{L}) + \mathcal{P}) = J \cap \mathcal{B} = J.$$

If $J \cap \mathcal{P} = \mathcal{P}$, then $J \supseteq \mathcal{P}$ and, since $J \supseteq \Phi(\mathcal{L})$, $J \supseteq \Phi(\mathcal{L}) + \mathcal{P} = \mathcal{B}$, hence $J = \mathcal{B}$. Consequently, there exists no subalgebras of \mathcal{B} properly contained between $\Phi(\mathcal{L}) + \mathcal{M}$ and \mathcal{B} , hence either $\Phi(\mathcal{L}) + \mathcal{M} = \mathcal{B}$ or $\Phi(\mathcal{L}) + \mathcal{M}$ is maximal in \mathcal{B} . If $\Phi(\mathcal{L}) + \mathcal{M} = \mathcal{B}$, then $\Phi(\mathcal{L}) + \mathcal{P}$ is not a reduced partial sum which is a contradiction. If $\Phi(\mathcal{L}) + \mathcal{M}$ is maximal in \mathcal{B} , then

$$\Phi(\mathcal{L}) + \mathcal{M}/\Phi(\mathcal{L}) \supseteq \Phi(\mathcal{B}/\Phi(\mathcal{L})) \cong (\mathcal{R} \cap \Phi(\mathcal{L})).$$

Hence $\mathcal{R} \subseteq \Phi(\mathcal{L}) + \mathcal{M}$. Since $\mathcal{M} \subseteq \Phi(\mathcal{L}) + \mathcal{M}$ and

$$x \in \mathcal{R} \cap \mathcal{P} \subset \mathcal{R} \subseteq \Phi(\mathcal{L}) + \mathcal{M},$$

$$\mathcal{P} = \{\mathcal{M}, x\} \subseteq \Phi(\mathcal{L}) + \mathcal{M}.$$

Then $\mathcal{B} = \Phi(\mathcal{L}) + \mathcal{P} \subseteq \Phi(\mathcal{L}) + \mathcal{M} \subseteq \mathcal{B}$ implies $\Phi(\mathcal{L}) + \mathcal{P}$ is not a reduced partial sum, a contradiction.

Hence $\Phi(\overline{\mathcal{B}}) = \mathcal{R}/\Phi(\mathcal{L}) = 0$ and (ii) is satisfied.

If $\mathcal{L}/\Phi(\mathcal{L})$ satisfies (ii), then $\pi(\Phi(\mathcal{A})) \subseteq \Phi(\pi(\mathcal{A})) = 0$ for every subalgebra \mathcal{A} of \mathcal{L} . Then $\Phi(\mathcal{A}) \subseteq \Phi(\mathcal{L})$ for every subalgebra \mathcal{A} of \mathcal{L} . \square

(Combining Proposition 2.2 and Theorem 2.1 we have:

Theorem 2.2. *Let \mathcal{L} be a Lie algebra such that \mathcal{L}' is nilpotent. Then $\mathcal{L} \in \mathbb{Q}$. \square*

Theorem 2.3. *Let $\mathcal{L} \in \mathbb{Q}$ and let φ be a Lie homomorphism of \mathcal{L} . Then $\varphi(\Phi(\mathcal{L})) = \Phi(\varphi(\mathcal{L}))$.*

Proof. $\varphi(\Phi(\mathcal{L}))$ is always contained in $\Phi(\varphi(\mathcal{L}))$. If $k = \text{Kernel } \varphi \subseteq \Phi(\mathcal{L})$, then equality holds. Suppose $\mathcal{P} \not\subseteq \Phi(\mathcal{L})$.

Let $\mathcal{L} = \mathcal{P} + k$ be a reduced partial sum. Using Lemma 2.2,

$$\Phi(\varphi(\mathcal{L})) = \Phi(\mathcal{L}/\mathcal{P}) \cong \mathcal{P} + \Phi(k)/\mathcal{P} = \varphi(\Phi(k)).$$

Since $\varphi(\mathcal{P} + \Phi(\mathcal{L})) = \varphi(\Phi(\mathcal{L})) \subseteq \Phi(\varphi(\mathcal{L})) = \Phi(\mathcal{L}/\mathcal{P}) = \varphi(\mathcal{P} + \Phi(k))$,

$$\mathcal{P} + \Phi(\mathcal{L}) \subseteq \mathcal{P} + \Phi(k) \subseteq \mathcal{P} + \Phi(\mathcal{L}).$$

Hence $\mathcal{P} + \Phi(\mathcal{L}) = \mathcal{P} + \Phi(k)$ and $\Phi(\varphi(\mathcal{L})) = \varphi(\Phi(k)) = \varphi(\Phi(\mathcal{L}))$. \square

Theorem 2.4. *Let $\mathcal{L} \in \mathbb{Q}$. Necessary conditions that an ideal \mathcal{J} of \mathcal{L} be the Frattini subalgebra of \mathcal{L} are the following:*

(i) $\Phi(\text{Ad}_{\mathcal{J}}(\mathcal{L})) = \text{Ad}_{\mathcal{J}}(\Phi(\mathcal{L}))$.

(ii) *There exists a subalgebra \mathcal{A} of \mathcal{L} such that $\mathcal{A}/\mathcal{J} \cong \text{Ad}_{\mathcal{J}}(\mathcal{L})/\text{Ad}_{\mathcal{J}}(\Phi(\mathcal{L}))$.*

Proof. (i) Let φ be the mapping from \mathcal{L} into the derivation algebra of \mathcal{J} defined by $\varphi(x) = \text{ad}_x$ restricted to \mathcal{J} for all $x \in \mathcal{L}$. Then

$$\varphi(\Phi(\mathcal{L})) = \text{Ad}_{\mathcal{J}}(\Phi(\mathcal{L})) = \Phi(\varphi(\mathcal{L})) = \Phi(\text{Ad}_{\mathcal{J}}(\mathcal{L})).$$

(ii) Let $\mathcal{B} = Z_{\mathcal{L}}(\Phi(\mathcal{L}))$. Suppose that $\mathcal{B} \not\subseteq \Phi(\mathcal{L})$ and let $\xi = \mathcal{L}/\Phi(\mathcal{L})$ and $\mathcal{R} = (\mathcal{B} + \Phi(\mathcal{L}))/\Phi(\mathcal{L})$. Since $\text{Ad}_{\Phi(\mathcal{L})}(\mathcal{L}) \cong \mathcal{L}/\mathcal{B}$ and

$$\text{Ad}_{\Phi(\mathcal{L})}(\Phi(\mathcal{L})) \cong \Phi(\mathcal{L})/Z(\Phi(\mathcal{L})) = \Phi(\mathcal{L})/\mathcal{B} \cap \Phi(\mathcal{L}) = (\mathcal{B} + \Phi(\mathcal{L}))/\mathcal{B},$$

$$\xi/\mathcal{R} \cong (\mathcal{L}/\Phi(\mathcal{L}))/(\mathcal{B} + \Phi(\mathcal{L})/\Phi(\mathcal{L})) \cong \mathcal{L}/(\mathcal{B} + \Phi(\mathcal{L})) \cong$$

$$\cong (\mathcal{L}/\mathcal{B})/((\mathcal{B} + \Phi(\mathcal{L}))/\mathcal{B}) \cong \text{Ad}_{\Phi(\mathcal{L})}(\mathcal{L})/\text{Ad}_{\Phi(\mathcal{L})}(\Phi(\mathcal{L})).$$

Since $\Phi(\xi) = 0$, there exists a subalgebra \mathcal{D} in ξ such that ξ is the reduced partial sum of \mathcal{R} and \mathcal{D} .

Using Proposition 2.2 and Lemma 2.1, $\mathcal{R} \cap \mathcal{D} \subseteq \Phi(\mathcal{D}) = 0$, hence $\mathcal{R} \cap \mathcal{D} = 0$.

Let \mathcal{H} be the subalgebra of \mathcal{L} which contains $\Phi(\mathcal{L})$ and corresponds to \mathcal{D} . Then

$$\mathcal{H}/\Phi(\mathcal{L}) \cong \mathcal{D} \cong \xi/\mathcal{R} \cong \text{Ad}_{\Phi(\mathcal{L})}(\mathcal{L})/\text{Ad}_{\Phi(\mathcal{L})}(\Phi(\mathcal{L})). \quad \square$$

If $\mathcal{B} \subseteq \Phi(\mathcal{L})$, then

$$\begin{aligned} Ad_{\Phi(\mathcal{L})}(\mathcal{L})/Ad_{\Phi(\mathcal{L})}(\Phi(\mathcal{L})) &\cong (\mathcal{L}/\mathcal{B})/(\Phi(\mathcal{L})/Z(\Phi(\mathcal{L}))) = \\ &= (\mathcal{L}/\mathcal{B})/(\Phi(\mathcal{L})/\mathcal{B} \cap \Phi(\mathcal{L})) = (\mathcal{L}/\mathcal{B})/(\Phi(\mathcal{L})/\mathcal{B}) \cong \mathcal{L}/\Phi(\mathcal{L}). \end{aligned}$$

Related to part (i) of Theorem 2.4, there are the following results.

Theorem 2.5. *Let $\mathcal{L} \in \mathbb{Q}$ and let \mathcal{J} be an ideal of \mathcal{L} containing $\Phi(\mathcal{L})$. Then, $\Phi(Ad_{\mathcal{J}}(\mathcal{L})) \cong Ad_{\mathcal{J}}(\mathcal{J})$ if and only if $\mathcal{J} = \Phi(\mathcal{L}) + Z(\mathcal{J})$. \square*

Theorem 2.6. *Let $\mathcal{L} \in \mathbb{Q}$ and let \mathcal{J} be an ideal of \mathcal{L} contained in $\Phi(\mathcal{L})$. Then, $\Phi(Ad_{\mathcal{J}}(\mathcal{L})) \cong Ad_{\mathcal{J}}(\mathcal{J})$ if and only if $\Phi(\mathcal{L}) = \mathcal{J} + Z_{\Phi(\mathcal{L})}(\mathcal{J})$. \square*

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Department of Mathematics and Informatics,
 "Mircea cel Bătrân" Naval Academy,
 Fulgerului 1
 8700 - Constantza
 Romania

